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DETERMINATION OF A REALISTIC ERROR BOUND FOR A CLASS OF IMPERFECT NONLINEAR CONTROLLERS

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I. INTRODUCTION

The synthesis of nonlinear controllers for nonlinear, time-varying plants according to the Direct Method of Liapunov has received considerable attention in the literature over the past several years. In all such controllers, corrective feedback is generated by means of an ideal signum function, the characteristics of which can only be approximated by some imperfect physical element. In as much as this imperfection could affect the stability of the controller, it becomes necessary to analyze the imperfect motion that could result. In particular it is desired to obtain conditions sufficient for such motion to be bounded and to find a realistic estimate of this bound in terms of design parameters. It is therefore intended that this study will not only add rigor to practical applications of the synthesis techniques but also increase their design flexibility. The existence of such a bound for some systems has been considered by Monopoli[2] however, the conservative nature of the bound rendered it ineffective as a design tool

In the following section the problem is formulated mathematically following a brief outline of a typical Liapunov synthesis technique.

In Section 3, specific imperfections are enumerated and it is shown that all give rise to a common state-space region outside of which the imperfect controller must behave as the ideal. This region of imperfect control is found to be essential to the subsequent bound development.

The next logical step is to investigate motion of the system in the region of imperfect control. This is carried out in Section 4 where the assumption of canonic form is made. A necessary and sufficient condition is found for imperfect motion to be bounded. This condition, obviously necessary for boundedness of motion not necessarily confined to the region, cannot, without further consideration, be shown to be sufficient.

Sufficiency is established in Section 5 after it is shown that the ideal controllers considered actually guarantee that the state vector approaches the switching plane monotonically. This proof opens the door to further investigation of the synthesis technique and, for purposes of this report, provides a necessary and sufficient condition for the bound, calculated in section 7.

In Section 6 some background is provided to justify the development in Section 5 and some design improvements are discussed.

Section 8 is comprised of two illustrative examples and Section 9 contains a discussion of the bound development to non-canonic systems.

2. PROBLEM STATEMENT

The problem considered herein pertains to single-input, single-output nonlinear, time-varying n^{th}-order control systems which can be represented mathematically by the following vector differential equation

$$\underline{\dot{x}} = A(t)\underline{x} + \underline{f}(\underline{x},u,t) + \underline{b}(\underline{x},u,t)\mathrm{Sgn}\{\gamma(\underline{x})\} . \tag{1}$$

Here $\underline{x}$, $\underline{f}$ and $\underline{b}$ are n-dimensional column vectors, A(t) is n x n and the scalar signum function, Sgn, having scalar argument , $\gamma(\underline{x})$, is defined as

$$\mathrm{Sgn}(\gamma) = \begin{bmatrix} +1 & ; & \gamma > 0 \\ \sigma & ; & \gamma = 0 \\ -1 & ; & \gamma < 0 \end{bmatrix} \tag{2}$$

where σ is defined within the limits

$$|\sigma| \leq 1 \tag{3}$$

Linear switching is assumed, thus

$$\gamma(\underline{x}) = \underline{P}^T\underline{x} = \sum_{i=1}^{n} p_i x_i . \tag{4}$$

Furthermore it will be assumed that $\underline{\dot{x}}$ is bounded for bounded $\underline{x}$.

The first two terms on the right of (1) represent the open-loop plant. The third term represents corrective feedback employing ideal switching through the signum function. This feedback, derived through synthesis procedures based on the Second Method of Liapunov or through some analogous procedure involving a Liapunov-type function, is assumed to guarantee that motion, $\underline{x}(t,\underline{x}_0 t_0)$, of (1) is asymptotically stable for all $(\underline{x}_0,t_0)\varepsilon R$.

The system (1) is therefore a general relay control system* employing linear switching which, under the assumption of ideal switching, is asymptotically stable. The synthesis of such Liapunov controllers is treated by Grayson[1], Monopoli[2], Lindorff[3], Taylor[4] and others and work along these lines is summarized by Grayson[5]. Characteristic of this synthesis technique is the definition of a positive-definite quadratic Liapunov function

$$V(\underline{x}) = \underline{x}^T P \underline{x} \tag{5}$$

the time derivative of which is maintained negative definite by controlling the sign of a certain term, $\underline{x}^T P(00\ldots f)$. This is accomplished by means of a corrective feedback signal appearing in f. This signal is given a magnitude sufficient to guarantee that its sign dominate the sign of f and it is given the sign of $-\underline{x}^T P(00\ldots 1)$. Of paramount importance in this technique is the fact that the sign-generating function must not alter the magnitude of the feedback signal. Therefore, ideal switching through the signum function is essential. However, since corrective feedback is formulated outside the plant and since any physical or electrical switching element can only approximate the ideal signum function characteristics, the designer must concern himself with the possible consequences of this approximation. Forced to use an element which may exhibit finite linear range, dead-zone or possibly hysteresis, the designer must analyze the motion of (1) that may result due to this imperfect control.

This analysis would apply also to cases wherein the designer may wish to deliberately incorporate imperfection into the switching element to conserve

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* The term "relay control" does not necessarily refer to the physical device but rather to its idealized characteristic, the signum function. It may well be that actual switching is achieved statically.

fuel or energy or to satisfy some such constraint.

Imperfect control may also result when additive noise enters into state measurement and thus affects the instrumented switching function argument, $\gamma(\underline{x})$.

It becomes apparent that in practical implementation, motion of the imperfect control system is no longer governed by (1) to be asymptotically stable. That is, the sign of $\dot{V}$ is no longer guaranteed negative definite. The problem considered herein is that of studying the imperfect control of systems which in idealized form are described by (1) and which have been designed via Liapunov's Direct Method. After obtaining sufficient conditions for the resulting motion to be bounded, the problem becomes that of obtaining a realistic estimate of this bound.

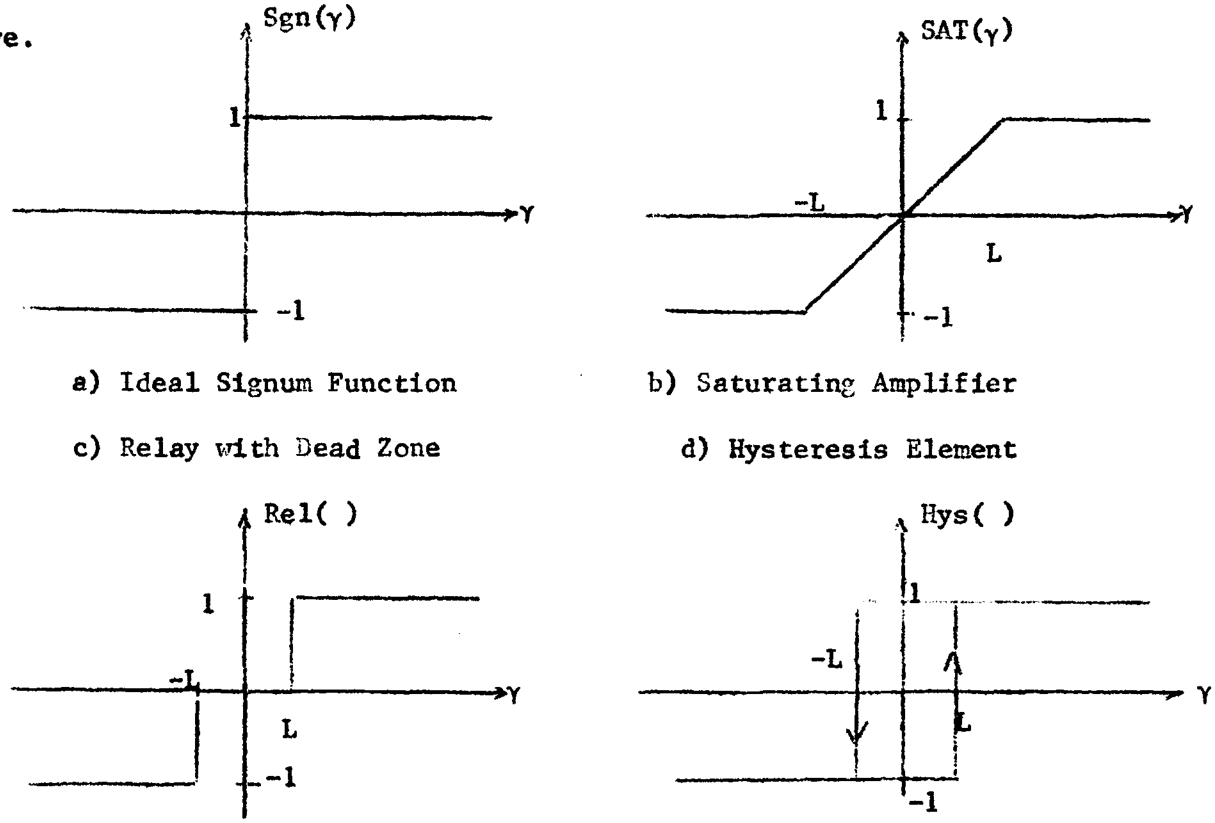
In the following section, controller imperfections are shown to give rise to a common state-space region outside of which motion of the imperfect controller is identical to that of the ideal controller. This is termed the region of imperfect control.

3. REGION OF IMPERFECT CONTROL

We now consider the imperfect control that results when the signum function, Fig. 1a, is approximated by some practical element such as a saturating amplifier, Fig. 1b, a relay with dead-zone, Fig. 1c, or hysteresis, Fig. 1d, or when bounded additive transducer noise enters into the measurement of the signum function argument, $\gamma(\underline{x})$. It is apparent in Fig. 1 that for $\gamma(\underline{x})$ of magnitude L or greater the approximating functions coincide with the ideal, thus motion is described by (1). However, for $\gamma(\underline{x})$ of magnitude less than L the approximations are poor in that they may have insufficient magnitude or incorrect sign. In this case motion is not governed by (1) and imperfect control results. It is important to note, however, that imperfect control can only result in the state-space region Ω defined by

$$\Omega = \{\underline{x}: \quad |\gamma(\underline{x})| < L\}. \tag{6}$$

This is termed the region of imperfect control for the imperfections enumerated above.



a) Ideal Signum Function

b) Saturating Amplifier

c) Relay with Dead Zone

d) Hysteresis Element

Figure 1

Signum Function and Approximations

This region of imperfect control is also common to other imperfections such as proportional transudcer noise, proportional transducer error, measurement delay and finite switching time. In the case of proportional noise and for the case of a percentage error of a transducer it follows that over any compact set in the state space a number L can be found that bounds the measurement imperfection in $\gamma(\underline{x})$. Thus the region of imperfect control is as described in (6).

A similar region results in the case of measurement delay and finite switching time. However, here, imperfect control is possible only in a certain time period, Δ, following the time that switching should occur, t_s. In as much as ideal switching occurs on the switching surface, $\gamma(\underline{x}) = 0$, and since $\dot{\underline{x}}$ is bounded for bounded $\underline{x}$ it follows that subsequent motion $\underline{x}(t,\underline{x}_s,t_s)$ for $t_s \leq t \leq t_s + \Delta$ is confined to some bounded region about $\underline{x}_s$ and the existence of such a number L is guaranteed.

Having established the state-space region in which imperfect control can result due to the imperfections considered, it is now helpful to investigate the possible motions of $\underline{x}$ in such a region.

4. BOUNDEDNESS OF SOLUTIONS IN THE REGION OF IMPERFECT CONTROL

It has been shown that motion of the imperfect control system is described exactly by (1) when the state vector $\underline{x}$ is in Ω', the complement of the region of imperfect control, Ω. Therefore, motions entirely contained in Ω' behave as though they were asymptotically stable provided they are also within R, the domain of asymptotic stability of (1). However, motion of relay control systems usually tends to the switching surface $\gamma(\underline{x}) = 0$ and thus to Ω. When $\underline{x}$ enters Ω imperfect control can result and asymptotic stability is no longer guaranteed. It is the purpose of this section to study the motion $\underline{x}(t,t_0,\underline{x}_0)$ under the assumption that $\underline{x}(t,t_0,x_0)\epsilon\Omega$ for all $t \geq t_0$. This will provide a necessary condition for bounded motion of the imperfect system throughout R and will lead the way for the bound calculation.

The general method of attack is to treat the constraint

$$\underline{x}(t,t_0,\underline{x}_0)\epsilon\Omega, \text{ for all } t \geq t_0, \tag{7}$$

which is equivalent to the constraint

$$|\gamma(\underline{x})| < L, \tag{8}$$

as a state-space constraint on one of the state variables x_i. This inequality constraint on x_i is then framed as an equality by introduction of a slack variable, $\alpha(t)$. Then substitution for x_i in (1) will result in a reduced-state system independent of the scalar input $u(t)$ but dependent on the slack variable $\alpha(t)$ which is treated as an input. Allowing $\alpha(t)$ to vary within its bounds, necessary and sufficient conditions for the boundedness of solutions of the reduced-state system are then found. These conditions are shown to be necessary and sufficient for the boundedness of solutions of the imperfect system throughout R.

The discussion up to this point has been general. However, since the technique does not apply to all systems in the form of (1) it is instructive to treat one special form and then discuss extension. It should be pointed out that the form treated is that particular form required of the synthesis techniques of references 1 through 4.

Assume (1) is in the Canonic Form, that is,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= m(t) \end{aligned} \tag{9}$$

where

$$m(t) = \sum_{i=1}^{n} a_i(t)x_i + f(\underline{x},u,t) + b(\underline{x},u,t)\mathrm{Sgn}\{\gamma(\underline{x})\}. \tag{10}$$

Linear switching is assumed,

$$\gamma(\underline{x}) = \sum_{i=1}^{n} p_i x_i \tag{11}$$

with the coefficients p_i constant. Considering motion confined to the region of imperfect control, Ω, defined in (6) which in this case is the hyperplanar region centered about the switching hyperplane, it follows that

$$|\gamma(\underline{x})| < L \tag{12}$$

or explicitly

$$\left|\sum_{i=1}^{n} p_i x_i\right| < L. \tag{13}$$

Rearranging terms and assuming $p_n > 0$, this may be framed as a constraint on x_n,

$$-\frac{1}{p_n}\sum_{i=1}^{n-1} p_i x_i - \frac{L}{p_n} < x_n < -\frac{1}{p_n}\sum_{i=1}^{n-1} p_i x_i + \frac{L}{p_n} \tag{14}$$

which is satisfied if

$$x_n = -\frac{1}{p_n}\sum_{i=1}^{n-1} p_i x_i + \alpha(t) \tag{15}$$

where $\alpha(t)$ varies in an unknown manner satisfying

$$|\alpha(t)| < L/p_n. \tag{16}$$

In light of this realization, the motion of $\underline{x}$ through Ω according to (1) is described by the following set of n-1 differential equations

$$\dot{x}_1 = x_2$$

$$\vdots$$

$$\dot{x}_i = x_{i+1}$$

$$\vdots$$

$$\dot{x}_{n-1} = -\frac{1}{p_n}\sum_{i=1}^{n-1} p_i x_i + \alpha(t) \tag{17}$$

along with the linear (15). In other words, motion of the n^{th} order, nonlinear, time-varying, single-input single-output differential system (9), subjected to the linear state-space constraint, (12), may be characterized by the $(n-1)^{st}$-order constant differential system with bounded input $\alpha(t)$. In as much as the subsequent calculation of an error bound is based solely on this <u>reduced-state</u> system, it is appropriate at this point to compare this <u>reduced</u> system (17) with the <u>original</u> control system, (9).

The reduced system characterizes the original system for the case $\underline{x} \epsilon \Omega$ in that it is capable of providing any motion that could be realized by the original system. This is accomplished by means of the slack variable $\alpha(t)$. From the definition of $\alpha(t)$, it follows that

$$\gamma(t) = p_n \alpha(t). \tag{18}$$

Thus motion of the reduced system with constant α corresponds to motion of the original system on a constant γ plane, and as α varies the γ-plane containing $\underline{x}$ varies. Therefore, motion of the original system, (9), in the region Ω is treated as motion parallel to the switching plane and motion perpendicular to the switching plane. It is by nature of this fact that the order of (17) is lower than that of (9).

It should be noted that some degree of conservativeness is present in the reduced system due to the fact that it is capable of motion that the original system cannot achieve. In particular, by placing no constraint on the time derivative of $\alpha(t)$, x_n is allowed to change instantaneously in the reduced system. For the original system, this is not possible in light of the assumption of bounded $\dot{\underline{x}}$.

Of importance here is the fact that the set of all possible solutions of the original system are contained in the set of all possible solutions of the reduced system. Thus boundedness of solutions of (9) would be implied by boundedness of solutions of (17). This will be considered in the following discussion.

Consider the vector representation of the reduced system (17)

$$\dot{\underline{x}}_r = A_r\underline{x}_r + \underline{b}_r\alpha(t) \tag{19}$$

where $\underline{x}_r$ is the reduced state vector, $(x_1, x_2, \ldots x_{n-1})^T$ and

$$A_r = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & \\ 0 & 0 & 1 & 0 & \ldots & \\ & & & & & 0 \\ & 0 & & & 1 & \\ & & & & 0 & 1 \\ -\frac{p_1}{p_n} & -\frac{p_2}{p_n} & & \ldots & & -\frac{p_{n-1}}{p_n} \end{bmatrix}, \quad \underline{b}_r = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \tag{20 a,b}$$

are (n-1) x (n-1) and (n-1) x 1 constant matrices respectively and $|\alpha(t)|$ is bounded by (L/p_n). Existence of a bound on solutions of this linear, time-invariant system with bounded input is guaranteed by classical stability theory. That is, all solutions are bounded if and only if A_r is a stability matrix.

Recalling that for the case $\alpha(t) = 0$, (19) represents motion on the switching plane, $\gamma(\underline{x}) = 0$, it is logical to require that such motion be stable, in the classical sense. This condition is necessary for the remaining discussion and it is considered to be valid for most systems in the form of (9). Notice, however, that, due to the conservative nature of the reduced system, the assumption on A_r has only been shown to be sufficient.

In light of this assumption, solutions of the reduced system are bounded and this implies that solutions of the original system, (9), $\underline{x}(t,t_0,\underline{x}_0)\varepsilon\Omega$ for all $t \geq t_0$, are bounded and 'escape' of $\underline{x}$ through Ω is ruled out. It will be shown, however, that this escape would occur for the case when A_r is an unstable matrix. Thus in terms of boundedness of solutions of the original system, the stability constraint is also necessary. Having established the above condition for boundedness of motion of the canonic system, (9), with linear switching, when such motion is confined to the region of imperfect control, Ω, we can now consider the unconfined motion of the imperfect system throughout the larger region R.

5. BOUNDEDNESS THROUGHOUT THE DOMAIN OF ASYMPTOTIC STABILITY

To summarize the development to this point, it has been shown by means of the reduced system that motion confined to Ω is bounded providing motion on the switching plane $\gamma(\underline{x}) = 0$ is bounded. This is guaranteed if the reduced system matrix A_r is a stability matrix. The problem to be considered in this section pertains to boundedness of motion in R.

Boundedness of motion confined to Ω for all $t \geq t_0$ is certainly necessary for boundedness in R but what is required here is a sufficiency condition. With the aid of Fig. 2 it will now be shown that boundedness of $\underline{x}$ in Ω does not immediately imply boundedness in R and thus further development is required.

Notice that motion $\underline{x}(t,t_0,\underline{x}_0)$ confined to Ω, Fig. 2, for all $t \geq t_0$ and any $\underline{x}_0 \epsilon \Omega$ according to (17) ultimately lie within some bounded region $B_\Omega \epsilon \Omega$. This follows from the stability matrix assumption on A_r. Response to the initial condition $\underline{x}_0(t_0)$ decays to the origin and response to the bounded input, $\alpha(t)$ is bounded. Therefore, any motion confined to Ω, originating inside B_Ω is contained in B_Ω for all subsequent time. However, consider a trajectory $\hat{\underline{x}}$ originating at $\underline{x}_0 \epsilon B_\Omega$ which is permitted to leave Ω. Corresponding to the point, a, at which $\hat{\underline{x}}$ leaves Ω is a Liapunov contour V_a inside of which it must remain when it is contained in $\Omega' \cap R$. From Fig. 2 it is evident that $\hat{\underline{x}}$ must return to Ω at some point b inside the V_a contour.

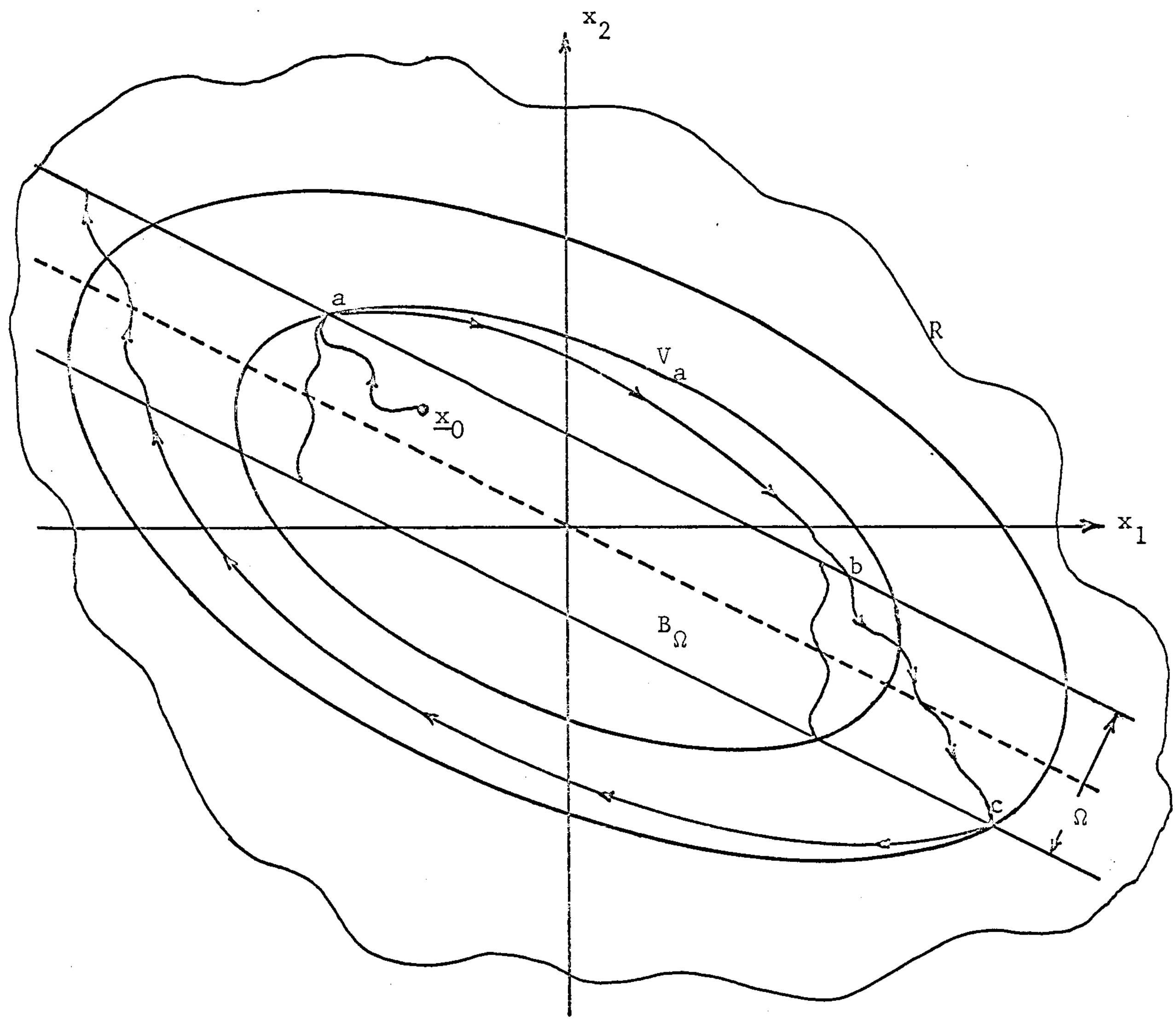


Figure 2

Escape by Cyclical Motion of $\underline{x}$

However, nothing at this point rules out the possibility of $\hat{\underline{x}}$ then leaving Ω at some point c outside V_a and continue in this fashion to escape! This discussion demonstrates that further development is required to show that the condition on A_r is necessary and sufficient to guarantee the boundedness of motion of the imperfect control system throughout R.

At this point, the class of systems which can be treated further would become quite limited, were it not for the fact that, in physical implementation of the Liapunov controller, choice of a Liapunov function in the synthesis procedure is reflected only in the coefficients p_i in the switching function. These terms are equal to the respective terms p_{ij} in the right hand column of P, the quadratic Liapunov function matrix, (5). Thus a particular set of terms $\{p_i\}$ could correspond to synthesis based on any one of several positive definite matrices P. In fact, there are an infinite number of such P's. The designer is therefore free to choose any P with the particular $p_{in} = p_i$, $i = 1, 2, \ldots n$ to analyze motion of the Liapunov controller. There is one particular matrix P, corresponding to the set $\{p_i\}$, which facilitates further development of the bound in R. This P matrix is positive semi-definite. The synthesis procedure based on a positive semi-definite Liapunov function is considered in the next section where it is found to have many advantages. It is helpful to summarize that discussion at this point.

For the class of Liapunov controllers which can be designed according to a positive semi-definite Liapunov function it follows that under the assumption of ideal switching, asymptotic stability can be achieved. In this case the Liapunov function $V(\underline{x})$ vanishes not only at the origin but at all points on a hyperplane in the state space. This hyperplane is necessarily the switching plane. Furthermore, negative semi-definiteness of $\dot{V}(\underline{x},t)$ assures that motion in the ideal system will approach the switching plane monotonically. However, as before imperfect control can occur inside Ω, (6), but with a semi-definite Liapunov function constant - $V(\underline{x})$ contours coincide with constant - $|\gamma(\underline{x})|$ contours. In as much as $\dot{V}(\underline{x},t)$ is strictly negative for $|\gamma| \geq L$ and thus for $V(\underline{x}) \geq V_L$, the corresponding Liapunov contour, it follows that once $\underline{x}$ enters Ω it cannot leave Ω. Therefore the bound obtained for the reduced system, (17), which represents this motion in Ω, is a valid bound for motion of the system in R. This bound is valid for any initial condition within R. Furthermore, it is shown that any system which reduces to stable A_r can be analyzed with positive semi-definite $V(\underline{x})$. The bound technique applies therefore to all such canonic systems represented by (9).

Thus for canonic systems employing linear switching the stability matrix constraint on the reduced system matrix, A_r, becomes necessary and sufficient to guarantee boundedness of the imperfect system. In the following section it is shown that this condition on A_r is implied by asymptotic stability of the ideal system, (1), assumed at the onset. In Section 7 the problem of calculating the bound will be considered.

6. DESIGN GENERALIZATION

In this section we consider systems designed from definite Liapunov functions as though they were designed from semi-definite functions. Justification for choosing semi-definite functions $V(\underline{x})$ and $\dot{V}(\underline{x},t)$ is based on the intuitive notion that if V vanishes on some manifold in $\{\underline{x}\}$ and is positive everywhere else and if $\dot{V}$ is negative except on the same manifold, where it vanishes, then V must decrease monotonically until, after a finite time, it reaches zero and remains zero for all time. These conditions therefore drive $\underline{x}$ to the manifold. More formally, the above notion is expressed in the following theorem attributed to LaSalle[6], which is modified for application to the problem at hand.

> Theorem 1: Let Ψ be a closed and bounded (compact) set in $\{\underline{x}\}$ with the property that every solution of (1) which begins in Ψ remains for all time in Ψ. Suppose there is a scalar function $V(\underline{x})$ which has continuous first partials in Ψ and is such that $\dot{V}(\underline{x},t) \leq 0$ in Ψ. Let E be the set of all points in Ψ where $\dot{V}(\underline{x},t)$ can vanish. Then every solution $\underline{x}(t,t_0,\underline{x}_0)$ with $\underline{x}_0 \epsilon \Psi$ approaches E monotonically as $t \to \infty$.

For the region Ψ we take the largest contour of the positive-definite Liapunov function which is entirely contained in R. The set E then becomes the intersection of the $\dot{V}(\underline{x},t) = 0$ set and Ψ. Solutions of (1) originating within Ψ therefore must approach E monotonically.

If $\gamma(\underline{x})$ and $V(\underline{x})$ are to vanish on the same plane then since $V(\underline{x})$ is quadratic it must be that

$$V(\underline{x}) = \tfrac{1}{2}K\{\gamma(\underline{x})\}^2 \tag{21}$$

or expressed in quadratic form

$$V(\underline{x}) = \tfrac{1}{2}K\underline{x}^T P\underline{x} \tag{22}$$

the elements of P must be

$$p_{ij} = p_i p_j. \tag{23}$$

The time derivative is then

$$\dot{V}(\underline{x},t) = K\gamma(\underline{x})\dot{\gamma}(\underline{x}). \tag{24}$$

The synthesis procedure in this case involves forcing sufficient magnitude of the control term appearing in $\dot{\gamma}$ so that its sign governs the sign of $\dot{\gamma}$ and in addition guaranteeing that $\dot{\gamma}$ not vanish except on $\gamma = 0$. The control term is then given the sign of $(-\gamma)$ and negative semi-definiteness of $\dot{V}$ is assured. The set E then is the intersection of the switching plane with Ψ. By LaSalle's theorem, the ideal system will attain E and therefore reach the switching plane $\gamma(\underline{x}) = 0$. If this system is to be asymptotically stable in some region R, then subsequent motion on the switching plane must be asymptotically stable. It was pointed out earlier that motion on the switching plane is described by the reduced-state system with $\alpha(t) = 0$, that is

$$\dot{\underline{x}}_r = A_r \underline{x}_r \tag{25}$$

where

$$A_r = \begin{bmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & & \\ 0 & & & & 1 \\ -\frac{p_1}{p_n} & -\frac{p_2}{p_n} & \cdot & \cdot & -\frac{p_{n-1}}{p_n} \end{bmatrix}. \tag{26}$$

This motion is asymptotically stable if and only if A_r is a stability matrix.

It becomes apparent that the designer is not free to treat every system as through it were designed with semi-definite V but rather only those systems having coefficients p_i which cause all (n-1) roots of A_r to have negative real part. However, it has just been shown that the class which can be treated is that class of systems which reduce to stable A_r and this is necessarily true of all systems which guarantee asymptotic stability inside some region R. Since this was assumed of the canonic system (9) we have general application.

In summary, it has been shown that motion of the Liapunov controllers considered in this report can be studied with semi-definite Liapunov functions. This implies that $\underline{x}$ approaches Ω monotonically and that once it reaches Ω, $\underline{x}$ cannot leave the region. Therefore the bound on $\underline{x}$ for motion in R is the same as the bound on $\underline{x}$ for motion confined to Ω. This bound may be found by solution of the reduced system, (19), and the linear constraint, (14).

Experience in applying the semi-definite design technique has pointed out several advantages over the conventional procedure. Besides offering a calculable bound, which in the next section is shown to be realistic, it lends itself to realistic estimation of convergence time both to the switching plane and to the bound. Furthermore, determination of the state-space region in which the control law, is valid, R, is greatly simplified. For these reasons, this modification to the design procedure of such Liapunov controllers is recommended.

7. BOUND CALCULATION

It has been shown that the bound on motion of the imperfect control system in R can be found by solving for the forced response of the reduced system

$$\dot{\underline{x}}_r = A_r \underline{x}_r + \underline{b}_r \alpha(t) \tag{27}$$

to the input $\alpha(t)$ bounded by

$$|\alpha(t)| \leq L/p_n \tag{28}$$

for the variables x_i; $i = 1, 2, \ldots n-1$ and by solution of

$$x_n = -\frac{1}{p_n}\sum_{i=1}^{n-1} p_i x_i + \alpha(t) \tag{29}$$

for the variable x_n. Calculation of such a bound can be readily accomplished in terms of the impulse response, $h_r(t)$, representing the relation between input $\alpha(t)$ and output $x_1(t)$ expressed in (27). From definition of h_r and (29) it follows that

$$x_i(t) = \int_0^t \alpha(t)\frac{d^{i-1}}{dt^{i-1}} h_r(t-\tau)d\tau, \quad i = 1, 2, \ldots n-1 \tag{30}$$

and

$$x_n = -\frac{1}{p_n}\sum_{i=1}^{n-1} p_i \int_0^t \alpha(t)\frac{d^{i-1}}{dt^{i-1}} h_r(t-\tau)d\tau + \alpha(t) \tag{31}$$

In light of the bound on $|\alpha(t)|$ a valid bound on $\underline{x}$ is $B = B_r \cup B_n$ where

$$B_r = \{\underline{x}_r : |x_i| \leq \frac{L}{p_n} \int_0^t \left|\frac{d^{i-1}}{dt^{i-1}} h_r(t-\tau)\right| d\tau,\ i = 1, 2, \ldots n-1\} \tag{32}$$

and

$$B_n = \{x_n : |x_n| \leq \frac{L}{p_n} \sum_{i=1}^{n-1} |p_i| \int_0^t \left|\frac{d^{i-1}}{dt^{i-1}} h_r(t-\tau)\right| d\tau + \frac{L}{d_n}\} \tag{33}$$

This general procedure has been outlined only for the purpose of completing this general development. It is very likely that with a specific problem in mind the designer may be able to determine a less conservative bound.

The conservative nature of this general cubical bound B results, in part, from application of the triangle inequality to obtain B_n and from the fact that although B_r is a valid bound for motion of the reduced system from the origin $\underline{x}_r = \underline{0}$, it contains points which cannot possibly be reached by the system. Examples point out that a more reasonable bound could be based upon the subset B_r' of B_r which includes all points in $\underline{x}_r$ which can be reached from the origin $\underline{x}_r = \underline{0}$ by application of an admissable control, $\alpha(t)$, $0 \leq t \leq \infty$, bounded according to (28). The set B_r' is thus the set of all points to which the reduced system can be controlled, from the origin. This 'region of controllability' is termed the 'reachable set',[7] conventionally defined for single-input, single-output, linear, time-invariant system as

$$B_r' = \{\underline{x}_r : \underline{x}_r = \int_{t_0}^t \exp(A_r(t-\tau))\underline{b}_r \alpha(\tau) d\tau;\ \alpha(t) \text{ admissable}\}. \tag{34}$$

In certain cases, it is possible to calculate the reachable set B_r' of the reduced-state system with admissable $\alpha(t)$ being defined by (12). From this set a bound may be determined in the form of some generalized norm. Examples will show this method to be less conservative than that outlined previously, but lacking generality. However, a general analog computer estimation technique will be outlined.

Calculation of the reachable set has been outlined[8] for an n^{th} order linear system having distinct real eigenvalues. After transforming such a system to diagonal form, a procedure is outlined for finding the boundary of the reachable set of the diagonal system. However, only in the case of first and second order systems, does this lead to an algebraic expression which may be transformed back to the original system. In as much as second-order reduced systems result from third-order control systems, the above results do have limited application. This has led the author to develop a closed-form solution for second-order systems having complex eigenvalues. These results are outlined below.

Framing this as an optimization problem incorporating bounded input, $\alpha(t)$, open endpoint and open time, wherein the terminal cost function directs the second-order reduced state system to achieve some extreme n-dimensional closed convex manifold, the solution derived via the Maximum Principle is, in fact, the minimum time solution with one arbitrary initial condition. Independent of this unspecified initial condition on one adjoint variable, the optimal forcing, $\alpha_0(t)$, is found to be a square wave of magnitude L/p_3, in accordance with (28) and frequency equal to the undamped natural frequency of the roots of the reduced-state system.

For reduced systems of orders greater than two with roots not necessarily distinct there is no known technique for calculating the reachable set. However, a very good approximation to the reachable set can be obtained by simulating the reduced system (19) on an analog computer. This is made possible by the fact that the reachable set is obtained only through maximum forcing of $\alpha(t)$. Observing any two states x_i, x_j, plotted in real time on an x-y plotter, scaling time and magnitude if necessary, an operator can readily learn to manipulate a full-force bang-bang control on $\alpha(t)$ to determine the region in the (x_i,x_j) plane which can be reached. This region is then the (x_i,x_j) projection of the reachable set. Repeating this procedure $\frac{n(n-1)}{2}$ times leads to all possible projections, with those involving the state x_n obtained by the additional simulation of (15). Therefore an approximation of the reachable set can be constructed from the $\frac{n(n-1)}{2}$ projections.

This approximation is found to be realistic in light of the fact that, for such systems as the reduced system, (28), convexity of the reachable set is guaranteed by convexity of the set of admissable control, (28)[7]. Therefore, 'holes' cannot occur.

Examples of this bound estimation technique are given in the following section.

8. EXAMPLES

In this section the bound development will be demonstrated on a second-order control system with transducer noise imperfection and a fourth-order system with hysteresis. These examples lead the way to discussion of extention of the technique to non-canonic systems.

Consider the ideal controller synthesized for a second-order plant with square-law damping[9]

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -Kbx_1 - a(x_2)^2 + \{c_1|x_1| + c_2|x_2| + c_3(x_2)^2\}\mathrm{Sgn}(x_1+2x_2). \qquad (35)$$

This control system in ideal form is asymptotically stable in a certain region R about the origin. However, due to noise encountered in the measurement of x_2 it is possible that imperfect control can result. This noise is known to be less than 1/3 volt, thus the region of imperfect control is limited to

$$\Omega = \{\underline{x}: |x_1 + 2x_2| < 2/3\}, \qquad (36)$$

sketched in Fig. 3.

For motion within Ω it follows that

$$-2/3 < x_1 + 2x_2 < 2/3 \tag{37}$$

By introducing the slack variable, $\alpha(t)$, this is satisfied if

$$x_2 = -\tfrac{1}{2}x_1 + \alpha(t) \tag{38}$$

where

$$|\alpha(t)| < 1/3. \tag{39}$$

By knowledge of the fact that $\underline{x}\varepsilon\Omega$, then, behavior of the state x_2 is estimated by (38) and we need not be concerned therefore with the second equation in (35). Substituting (38) into the first equation gives

$$\dot{x}_1 = -\tfrac{1}{2}x_1 + \alpha(t). \tag{40}$$

This is the reduced system. In this case no elaborate technique is required for calculating the bound on x_1. It results that

$$|x_1| < 2/3 \tag{41}$$

and from (38) and (39)

$$|x_2| < 2/3. \tag{42}$$

The bound on motion of the imperfect system is therefore the intersection of (41) and (42) and Ω, sketched in Fig. 3.

This bound can be reasoned from the phase-variable structure of (35). In as much as $\dot{x}_1 = x_2$ then when x_2 is positive x_1 must be increasing and when x_2 is negative x_1 must be decreasing. This implies that motion confined to Ω in the first and second quadrant of the phase plane must eventually lead to the third or fourth quadrant and visa-versa. Furthermore, once $\underline{x}$ passes through the line $x_1 = 2/3$ or $x_1 = -2/3$ it can never go back through. This establishes the same bound as above.

The next example is a controller which, in ideal form, guarantees asymptotic stability of a fourth-order, dynamically-unstable plant. The plant

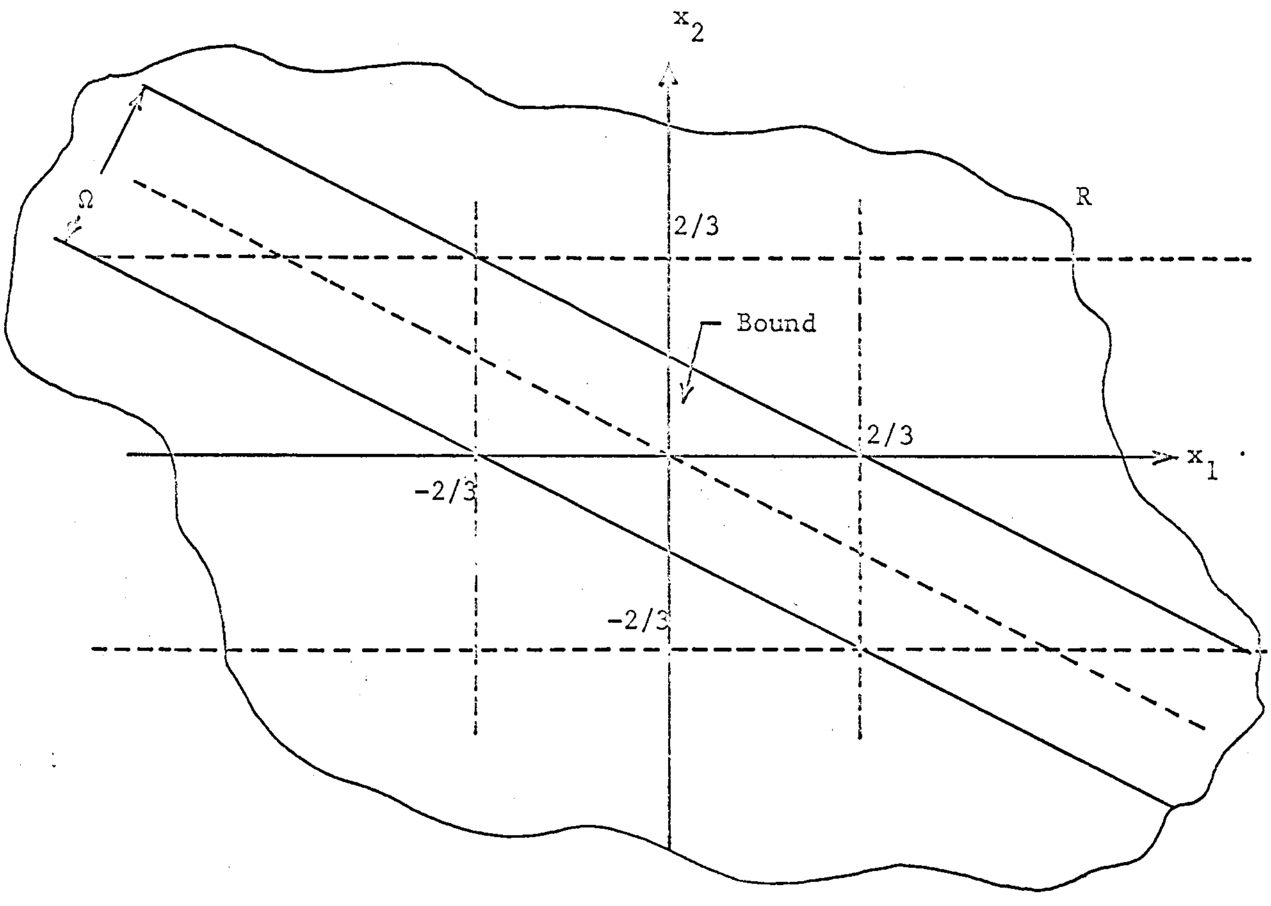


Figure 3

Example of a Bound Estimate

consists of a rigid, inverted pendulum mounted on a positioned cart, as sketched in Fig. 4. In accord with the design in reference 10 the assumption of small pendulum angle, ϕ, leads to linearization of the equations of motion which are then transformed into the canonic form

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \lambda^2 x_3 - \dot{s} + 3\mathrm{Sgn}(x_1 + 3x_2 + 4x_3 + 2x_4),
\end{aligned}
\tag{43}
$$

where $\dot{s}$ is a bounded term. This system was simulated on the PACE 231-R analog computer. A hysteresis function, as described in Fig. 1d, with $L=1/10$, was substituted for the ideal signum function. The system limit-cycled and this was recorded on several x-y plots to be compared with the theoretical bound developed below.

In this case motion in the region of imperfect control can be represented by

$$x_4 = -1/2x_1 - 3/2x_2 - 2x_3 + \alpha(t), \tag{44}$$

and the reduced system

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= -1/2x_1 - 3/2x_2 - 2x_3 + \alpha(t)
\end{aligned}
\tag{45}
$$

where

$$|\alpha(t)| < 1/20. \tag{46}$$

To obtain an estimate of the bound, (44) and (45) were simulated on the analog computer as discussed in the previous section. The results are found in Figures 5a, b, c to bound the limit cycle that occurred. In fact this bound

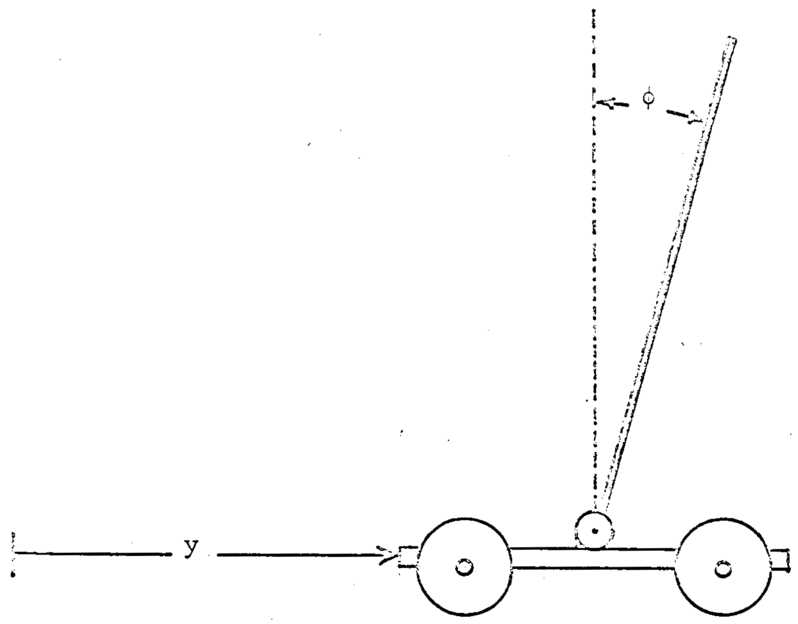


Figure 4

A Dynamically Unstable System

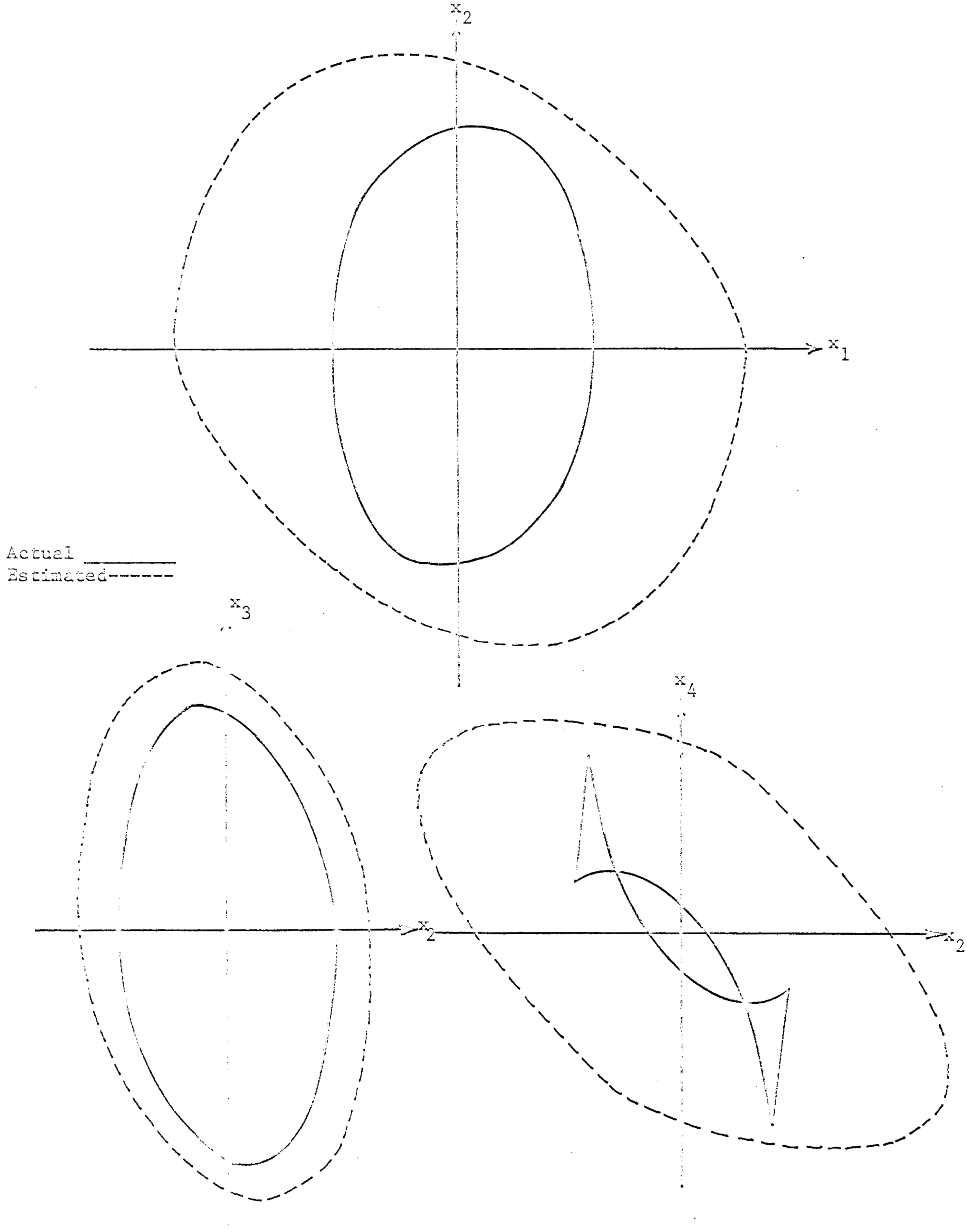


Figure 5

Comparison of Actual Bound with Estimated Bound is quite realistic when compared with any other estimate found in the literature[2].

Although these examples involve a numerical bound it follows that the technique would yield an algebraic bound if the problem were framed in that manner. It is this property that renders the development useful for design purposes. Noteworthy also is the fact that the bound approaches zero with L.

With these examples in mind, it is now possible to discuss extention of the technique to non-canonic systems.

9. EXTENSIONS

It was pointed out in Section 4 that the canonic form was required of the synthesis procedures of references 1 through 4. However, this requirement can be relaxed by employing semi-definite rather than definite functions for V and $\dot{V}$. It therefore becomes important to consider extension of the bound development to non-canonic systems.

We now return to the general equation (1).

$$\underline{\dot{x}} = A(t)\underline{x} + \underline{f}(\underline{x},u,t) + \underline{b}(\underline{x},u,t)\mathrm{Sgn}\{\gamma(\underline{x})\} \qquad (47)$$

and recall that the assumption of canonic form, (9), implies that A(t) be of the form

$$A(t) = \begin{bmatrix} 0 & 1 & 0 \cdots\cdots\cdots \\ 0 & 0 & 1 \quad 0 \quad \cdots \\ 0 & 0 & 0 \quad 1 \quad 0 \\ \cdot & \cdot & \cdots\cdots \\ \cdot & \cdots\cdot & \cdots\cdots\cdot \ 1 \\ a_1(t) & a_2(t) & a_3(t) \ . \ . \ a_n(t) \end{bmatrix} \qquad (48)$$

and that $\underline{f}$ and $\underline{b}$ possess only one non-zero term which must be in the n^{th} row. The primary reason that the bound development applies to this form is that the expression for x_n (15),

$$x_n = -\frac{1}{p_n}\sum_{i=1}^{n-1} p_i x_i + \alpha(t) \tag{49}$$

resulting from the assumption $\underline{x}\epsilon\Omega$, can be substituted into (47) thus reducing the order of this vector differential equation by one and resulting in a linear time-independent equation having bounded input, $\alpha(t)$. Solutions of this equation are not difficult to study due to the absence of nonlinearities, time-variation and possibly unknown terms. It is straight-forward to see that the same would be true if $\underline{f}$ and $\underline{b}$ had non-zero terms in the i^{th} row only. In this case the condition $\underline{x}\epsilon\Omega$ is solved for x_i. This would have the form

$$x_i = -\frac{1}{p_i}\sum_{\substack{j=1 \\ j\neq i}}^{n} p_i x_i + \alpha(t) \tag{50}$$

where $\alpha(t)$ would be limited by

$$|\alpha(t)| < L/p_i \tag{51}$$

This extension could be carried out only if $p_i \neq 0$.

It is also possible to extend the discussion to treat non-canonic matrices $A(t)$. However, this allows time-variable coefficients to enter into the reduced-system matrix $A_r(t)$ and they would have to be considered in the bound estimation. Such terms could affect conservativeness of the estimate.

Extension is also possible to systems which have more than one non-zero term in $\underline{f}$ and $\underline{b}$ but qualifications become so tedious that this is not considered further.

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